Engineering Notes

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Compressible Flow in a Long Tube Closed at One End

M.D. Bennett* and F.V. Wyatt†
Sandia National Laboratories
Albuquerque, New Mexico

Introduction

Measurements of the presures acting on missiles are usually made during a period of unsteady flight. If the measured quantity changes rapidly or if the capillary tube that connects the body orifice to the sensing element is long, the pressure differential which develops along the tube can become large. Consequently the gauged value at one end may be siginficantly different from the applied value at the other end. To intepret information of that type, the transient pressure response of the measuring system must be determined.

Ducoffe[1] and later Sato[2] investigated both theoretically and experimentally the pressure stabilization time in pneumatic lines subjected to the step-type forcing functions that often arise in wind-tunnel tests. Their models involved a long tube that joined the orifice to a cavity, which represented the void volume associated with a pressure transducer, and the resulting differential equation was numerically solved.

The analysis described here concerns the pressure variations in a line that terminates at a sensing instrument of negligible volume, a geometry which can be compositely represented by a long tube closed at one extremity. An elementary derivation of the response equation that evolves from the laminar flow of a gas moving at subsonic speed is given, along with a closed-form solution for the limiting case of slowly-varying pressure gradients. It is found that after a transient phase the difference in the pressures acting at the ends of the duct is proportional to the coefficient of fluid viscosity μ, the square of the tube length-diameter ratio L/d, and the time rate of change of the applied pressure p_0, and is inversely proportional to the magnitude of the pressure. Comparisons with numerical evaluations of the governing equation establish the range of applicability of the solution for flights in exponential atmospheres.

Pressure Response Equation

Most of the exact analytical solutions to viscous compressible fluid-dynamic problems involve one spatial coordinate only, and the present inquiry, with the asymptotic solution, concerns that category. Beginning with the fully-developed flow of a compressible fluid in a straight tube of constant diameter, the one-dimensional continuity equation is

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}=0 \tag{1}$$

where ρ represents the fluid density, t denotes time, and x is the axial station along the tube. Although the delineation of the fluid velocity u implicates two spatial dimensions (radial and axial), an equivalent one-coordinate rate will be utilized in a manner such that the account of mass exchange remains unaltered. Therefore, $u=u(t,x)$ here signifies a time-dependent average local velocity measured with respect to the moving body. Since the tube by decree has been made long, the flow process will be essentially isothermal if the tube heat capacity is large. It follows that when the fluid behaves in accordance with the perfect gas law, the density description involves one spatial direction also. Hence, $\rho=\rho(t,x)$ in Eq. (1).

Combining the unsteady continuity equation with the state equation ($p\sim\rho$) yields

$$\frac{\partial p}{\partial t}+\frac{\partial p u}{\partial x}=0 \tag{2}$$

where $p=p(t,x)$ is the pressure to be determined.

To express the average velocity called for in Eq. (2), a somewhat detailed view of the fluid motion is needed. The Navier-Stokes equations are introduced, with the flow temporarily taken to be steady and incompressible, and with the ordinarily negligible body forces ignored. These restrictions, together with the previous assumptions, eliminate two of the motion equations. The remaining statement details the balance of the pressure induced force that impels the fluid and the viscous restraining force that arises from the laminar shear stress. In cylindrical coordinates,

$$\frac{\mathrm{d}p}{\mathrm{d}x}=\frac{1}{r}\,\frac{\mathrm{d}(\mu r\mathrm{d}u/\mathrm{d}r)}{\mathrm{d}r} \tag{3}$$

where r is the radial distance from the center of the circular section. Imposing the usual zero-velocity-at-the-wall condition, the constant-viscosity solution to Eq. (3) takes the form $u=(1/4\mu)(r_0^2-r^2)\mathrm{d}p/\mathrm{d}x$, which is the well known Hagen-Poiseuille pipe flow formula. The equivalent uniform velocity for the parabolic profile is

$$u=\frac{2}{r_0^2}\int_0^{r_0} ur\mathrm{d}r=\frac{-r_0^2}{8\mu}\,\frac{\mathrm{d}p}{\mathrm{d}x}$$

where r_0 denotes the pipe inner radius. Applying this relationship over infinitesimal lengths of the tube, the average local velocity in a compressible flow becomes

$$u=-\frac{r_0^2}{8\mu}\,\frac{\partial p}{\partial x} \tag{4}$$

Then, by assuming a state of quasi-steady flow and substituting Eq. (4) into Eq. (2), the pressure response equation is obtained in the form

$$\partial p/\partial t=(r_0^2/8\mu)\;[(\partial p/\partial x)^2+p\partial^2p/\partial x^2]$$

which arranges to

$$\frac{\partial p}{\partial t}=\frac{k\partial^2 p^2}{\partial x^2} \tag{5}$$

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*Member of Technical Staff, Aerodynamics Department. Associate Fellow AIAA.

†Member of Technical Staff, Aerodynamics Department.

where $k = r_0^2/16\mu$. For a tube with the open end located at station $x = 0$, the required boundary and initial conditions are $p(t,0) = f(t)$, $\partial p/\partial x(t,L) = 0$, and $p(0,x) = f(0)$; the second of these stipulations is inferred from Eq. (4) by specifying that the fluid velocity is zero at the tube's closed end ($x = L$). Ducoffe[1] and, by way of a rigorous mathematical foundation, Sato[2] both obtained the identical diffusion-type behavior conveyed by Eq. (5), but with appropriately different boundary conditions.

Equation (5) can be made dimensionless by defining a unitless length, pressure, and time: $X \equiv x/L$, $p \equiv p/p^*$, and $T \equiv t/t^*$, respectively, where the asterisk indicates any suitable constant reference quantity. Changing terminology from tube radius to tube diameter d, the end product, an alternate version of the second-order partial-differential response equation, becomes

$$\frac{\partial P}{\partial T} = \frac{\alpha \partial^2 P^2}{\partial x^2} \tag{6}$$

where the coefficient is $\alpha = p^* t^* (d/L)^2 / 64\mu$. The transformed conditions are $P(T,0) = F(t)$, $\partial P/\partial X(T,1) = 0$, and $P(0,X) = F(0)$, and the new functions are $F(T) = f(t)/p^*$, $\partial P/\partial X(T,1) = (L/p^*)\partial p/\partial x(t,L)$, and $F(0) = f(0)/p^*$. Equation (6) is of a type that also applies to flow through porous media, and to heat conduction in materials with a variable diffusivity—one that maintains proportionality to the square root of temperature. There is no general solution.

Limiting Solution

Equations (5) and (6), which are nonlinear in derivatives with respect to x, concern the pressure history in a capillary that is being either slowly charged with a compressible fluid or discharged, and the analytical solution addresses that domain of validity. When the variation of the input pressure is not excessive, the time derivatives of the pressures at all stations ultimately will have nearly the same value. If the derivative $\partial p/\partial t$ is taken to be independent of x, a boundary-value problem replaces the original initial-volume problem, in which case the response equation becomes linear with respect to the square of the dependent variable. With that arrangement, Eq. (5) reduces to

$$\frac{\partial^2 p^2}{\partial x^2} = \frac{1}{k} \left.\frac{\partial p}{\partial t}\right|_{x=0} \tag{7}$$

where $(\partial p/\partial t)|_{x=0} \equiv \mathrm{d}p_0/\mathrm{d}t$, the time derivative of the known pressure at the tube entrance. Integrating twice with respect to x and insetting the boundary conditions yields the axial pressure distribution, $p^2 = p_0^2 - 64\mu\dot{p}_0\ (L/d)\ (x/d) + 32\mu\dot{p}_0(x/d)^2$, from which the pressure p_1 at the tube's closed end is found to be

$$p_1 = [p_0^2 - 32\mu\dot{p}_0(L/d)^2]^{1/2} \tag{8}$$

where $p_0 = f(t)$ is any continuous function having time derivatives, and $\dot{p}_0 \equiv \mathrm{d}p_0/\mathrm{d}t$. A useful approximation to Eq. (8) can be obtained since, consistent with the assumption of small gradients, one of the factors that appears in the development of the formula is subject to minor modification: $p_0^2 - p_1^2 = (p_0 + p_1)(p_0 - p_1) \cong 2p_0(p_0 - p_1)$. The effect is reflected in the final expression $p_0 - p_1 = 16\mu(L/d)^2\dot{p}_0/p_0$, which readily describes the pressure differential acting over the tube's length during periods of unsteady flow.

These asymptotic solutions obviously exclude a portion of the account of transient events since, for example, when a driving pressure ceases to vary, the predicted pressure at the opposite end of the duct instantaneously acquires the asymptotic value; i.e., $p_1 = p_0$ if $\dot{p}_0 = 0$, a behavior that traces reality only after an elapsed interval of time. However, the associated error is small if $16\mu(L/d)^2\dot{p}_0 \ll p_0^2$.

Results

Equation (8) is compared with numerical evaluations of Eq. (6) in Fig. 1. Free-stream pressure $p_0 = p_i e^{-ct}$ acts at the tube's open end, and p_1 is the resultant pressure at the closed end; p_i denotes the initial ambient pressure, which is applied throughout the tube at time zero, $t = 0$. The dimensionless parameter $\alpha = |(p_i/c)(d/L)^2/64\mu|$ involves the missile trajectory p_i/c, tube length/diameter ratio L/d, and gas viscosity μ, and it characterizes the pressure variations that arise theoretically from a viscous flow, with an assumed fully-developed laminar velocity profile. Although the graph extends over only one-unit scaled time, the closed-form asymptotic solution and the numerical results are essentially identical for all indicated α's when $|c|t > 1$.

The forcing function selected for the illustration, $f(t) = p_i \exp(-ct)$ where $p_i = p^*$ denotes the initial ambient pressure, describes the freestream pressure during ascending ($c > 0$) or descending ($c < 0$) flight at constant velocity in an isothermal atmosphere; the time constant $|c| = 1/t^*$, which derives from the transformation $T = |c|t$, is fixed by the known rate of change of altitude. It can be observed that when the system coefficient $\alpha = |(p_i/c)(d/L)^2/64\mu|$ is of moderate magnitude, the analytic approximation approaches the numerical solution after an initial transient phase has passed. With large magnitude, the agreement during the transient period markedly improves, so that when the gradients in the tube are indeed small, the closed-form solution is essentially in accord at all times. In any event, the pressure discrepancy, which is maximum at time zero, remains less than 9% if $\alpha > 3$, and it monotonically decreases as time advances.

Discussion

While no experiments have been performed here to assess the accuracy of these results, it is noted that steady incompressible flows in commercial pipes maintain a laminar velocity profile at Reynolds numbers up to about 2300. In the present case of unsteady flow, the local Reynolds number can be calculated by combining equations developed previously. According to Eq. (4), along with the derivative of the axial-pressure-distribution expression, the largest velocity in the assumed fully-developed flow occurs at the open end of the tube, and is independent of the tube diameter at that particular station. Its average value, which is one half the

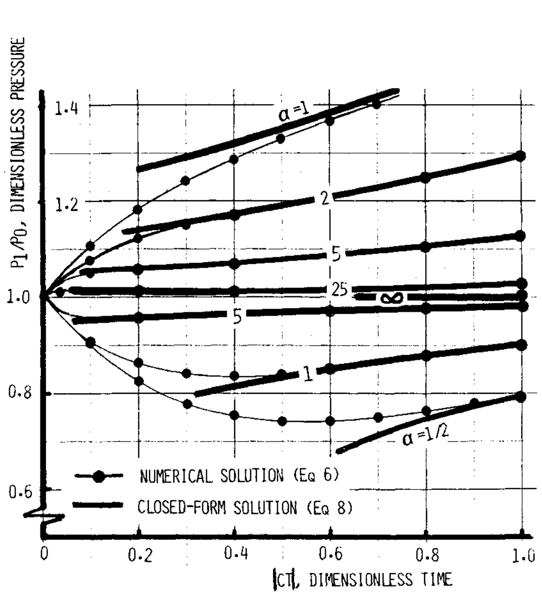

Fig. 1 Pressure response in a capillary tube during ascending ($p_1/p_0 > 1$) or descending ($p_1/p_0 < 1$) flight in an exponential atmosphere.

maximum, amounts to $u_0 = L/(p_0/\dot{p}_0)$. The corresponding Reynolds number, which also is maximum at the inlet station, is a function of the tube length-diameter product: $R_N = |\dot{p}_0 Ld/(\mu RT)|$, where R and T are the gas constant for air and the tube temperature, respectively. Both the Reynolds number and the fluid velocity at the tube entrance maintain proportionality to the rate of change of the applied pressure, but the Reynolds number, unlike the velocity, does not depend explicitly on the pressure level.

The tube entrance length required to produce the assumed state of fully-developed flow is dependent on Reynolds number. Boussinesq, as reported by Daily and Harleman,[3] obtained theoretically a length estimate for steady laminar flow that agrees well with experiment: $(L/d)_{\text{entrance}} = 0.065\ R_N$. In the present work, the tube is taken to be long compared with this dimension.

Acknowledgment

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Drag Reduction of Perforated Axisymmetric Bodies in Supersonic Flow

Eugen Serbănescu* and George Savu*
The National Institute for Scientific and Technical Creation
Bucharest, Romania

Nomenclature

B	$=\sqrt{M^2-1}$
C_{Dp}	= pressure drag coefficient
C_p	= pressure coefficient
$f(x)$	= porosity function
L^*	= maximal axial dimension of the body
M_∞	= freestream Mach number
p, p_t	= nondimensional state and total pressures, p^*/p^*_∞ and p^*_t/p^*_∞, respectively
Re	= reference Reynolds number
S	$=\pi r^2_{\max}$
U_∞	= freestream velocity
u, v	= nondimensional axis and radial perturbance velocities, u^*/U^*_∞ and v^*/U^*_∞, respectively
v_n	= nondimensional normal surface velocity, v^*_n/U^*_∞
ϵ	= thickness ratio of parabolic-arc spindle, $2r^*_{\max}/L^*$
θ^*	= cone half-angle
ρ	= nondimensional fluid density, ρ^*/ρ^*_∞
σ	= nondimensional porosity factor, $\sigma^* U^*_\infty/p^*_\infty$

Superscripts and Subscripts

( )*	= dimensional parameter
( )′	= derivative
∞	= undisturbed conditions
e	= equivalent solid body
max	= maximal
p	= plenum chamber

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*Research Scientist, Aerodynamics Department.

Introduction

THE optimization of aerodynamic body profiles relating to drag reduction has become a current effort in aerodynamics. The well-known shapes of the minimum drag bodies of revolution (Sears-Haack body, von Kármán ogive, Lighthill body, etc.) reach their optimum at a single flight Mach number, the one called "on-design."

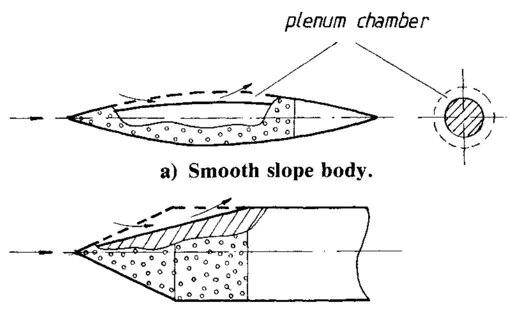
The idea we propose in the present Note leads to an aerodynamic drag reduction not only of the "on-design" point, but also "off-design" points, based on the use of perforated surfaces. As a matter of fact, a numerical treatment of a *self-adapting aerodynamical system* useful for axisymmetric bodies is presented. This implies both massive suction and massive blowing through the axisymmetric perforated surface. The theoretical framework (large rate of mass transfer through perforated surfaces) is the same as earlier research.[1-4] However, in our proposal, the existence of a cavity (plenum chamber) within the perforated surface is considered, so that a secondary (plenum) flow could be born due to the difference between the pressures acting "naturally" on the connected zones of the body (Fig. 1). No appropriate installation, device, or pump is imposed. *The system works itself.* As a consequence, a new pressure distribution will occur as the result of the interaction between the outer (main) and inner (secondary/plenum) flow. This interaction is allowed and conrolled by the distribution and inclination of the perforated surface holes as well as by the size of the holes to the cavity wall thickness ratio. (All of these parameters are incorporated into the porosity coefficient σ).

In this Note the normal velocity on the perforated surface is no more imposed, as in other works,[1-4] but is the result of the above aerodynamic mechanism and governed by the Darcy law (accepted in the mass transfer through the perforated walls case).[5]

A point of view similar to Refs. 1-4 has been adopted regarding the principal effect of the massive mass transfer which is considered inviscid because the normal velocity v_n can be much larger than the usual order of magnitude (0.0001-0.01) required for the consistency of boundary-layer theory,[6] i.e.,

$$v_n \gg Re^{-1/2}$$

The physical phenomenon is similar to that which takes place in wind tunnel test section with perforated walls, where the viscous effect is also negligible.



a) Smooth slope body.

b) Cone-cylinder ogive.

Fig. 1 Bodies of revolution with a perforated surface.